Discipline: **Physics**

Subject: **Electromagnetic Theory**

Unit 26:

Lesson/ Module: **Lienerd – Wiechert Potentials**

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Learning Objectives:

From this module students may get to know about the following:

1. *The general nature of the wave equation*
2. *Green's function method of solving inhomogeneous wave equations; the solution of the wave equation for the electromagnetic field.*
3. *Liénard-Wiechert potentials obtained as solutions of the wave equation.*
4. *The electromagnetic field tensor obtained from these potentials.*
5. *The explicit expressions for the electric and the magnetic field obtained from the field tensor.*

26. Liénard-Wiechert Potentials

26.1 Introduction – The Wave Equation

In this module we consider the fields produced by a single charged particle, or in fact a beam of particles moving along a given trajectory. This problem is of considerable interest and of practical importance in the context of high energy accelerators. In these machines a beam of particles are accelerated to extremely high energies and then used to perform scattering experiments. This is one of the main experimental techniques in the study of high energy nuclear and particle physics. A particle or a beam of particles moving along a path constitutes a current which acts as a source of electric and magnetic fields. The fields produced have the nature of radiation fields leading to the dissipation of energy by radiation. To study the fields and radiation produced by such accelerated charges we have to solve the inhomogeneous Maxwell equations. This is what we proceed to do now. Since the speeds involved are sometimes ultrarelativistic (in fact that is the case of real interest to us here), we begin with a covariant treatment of the problem.

26.1.1 The Wave Equation

The electromagnetic fields $F^{\alpha\beta}$ arising from an external source $J^{\alpha}(x)$ satisfy the inhomogeneous Maxwell equations

$$\partial_{\alpha} F^{\alpha\beta}(x) = \mu_0 J^{\beta} \qquad (1)$$

Here *x* is the four-vector $(x^0 = ct, \vec{x})$, $J^{\alpha}(x)$ the current four-vector $(J^0 = c\rho, \vec{J})$, ∂_{α} the four-divergence $(\frac{\partial}{\partial x^0}, \vec{\nabla})$ and $F^{\alpha\beta}$ the field tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \qquad (2)$$

With the definition of the fields in terms of the four-potential $A^{\alpha}(x) = (A^0 = \Phi / c, \vec{A})$:

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha} \quad (3)$$

the equations take the form

$$\Box\, A^{\beta} - \partial^{\beta}(\partial_{\alpha} A^{\alpha}) = \mu_0 J^{\beta} \quad (4)$$

If the potentials satisfy the Lorentz gauge condition, $\partial_{\alpha} A^{\alpha} = 0$, then they are solutions of the wave equation

$$\Box\, A^{\beta} = \mu_0 J^{\beta} \quad (5)$$

26.2 Green's Function for the Wave Equation

To solve this inhomogeneous differential equation, we construct a Green's function

$$\Box\, D(x, x') = \delta^{(4)}(x - x') \quad (6)$$

Here $\delta^{(4)}(x - x') = \delta(x_0 - x'_0)\delta(\vec{x} - \vec{x}') = \frac{1}{c}\delta(t - t')\delta(\vec{x} - \vec{x}')$ is the four dimensional delta function. The solution for the potential is then given by

$$A^{\alpha}(x) = A^{\alpha}{}_{I}(x) + \int d^4x' D(x - x') J^{\alpha}(x') . \quad (7)$$

The term $A^{\alpha}{}_{I}(x)$ is a solution of the corresponding homogeneous wave equation and has to be adjusted to obtain the solution with proper boundary conditions. In the absence of boundary surfaces, the Green's function can involve only the absolute distance between the points. Thus if $z = x - x'$ we seek solution to the equation

$$\Box\, D(z) = \delta^{(4)}(z) \quad (8)$$

There are several ways we could go about solving this equation. They are all equivalent at some level or other. We use the method of *Fourier transforms*.

The four dimensional Fourier transform of the desired Green's function is defined by

$$D(z) = \frac{1}{(2\pi)^4} \int d^4k \tilde{D}(k) e^{-ikz} \quad (9)$$

where $k.z = k_0 z_0 - \vec{k}.\vec{z}$. On operating by four-dimensional Laplacian $\Box$ on both sides, we obtain

$$\Box D(z) = \frac{1}{(2\pi)^4}\int d^4k\tilde{D}(k)(-k.k)e^{-ikz} \tag{10}$$

The four dimensional delta function has the representation

$$\delta^{(4)}(z) = \frac{1}{(2\pi)^4}\int d^4k e^{-ikz} \tag{11}$$

On substituting equations (8) and (9) into (6) and equating the integrands on the two sides we obtain

$$\tilde{D}(k)(-k.k) = 1 \Rightarrow \tilde{D}(k) = -\frac{1}{k.k} \tag{12}$$

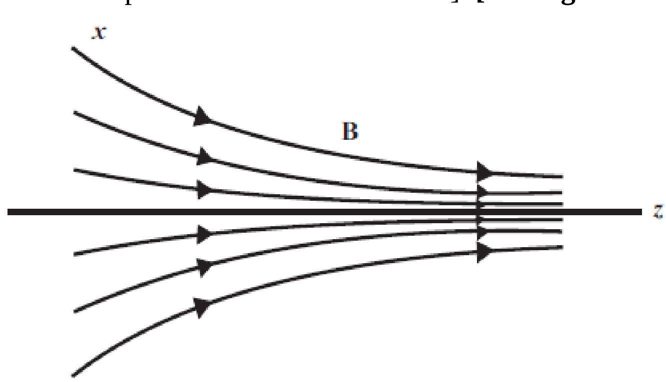
We therefore know that the Green's function has the form

$$D(z) = \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{-ikz}}{k.k} \tag{13}$$

The integrand in this expression is singular when $k.k = k_0{}^2 - \vec{k}^2 = k_0{}^2 - \kappa^2$ ($\kappa = |\vec{k}\,|$) vanishes. Recall that the presence of singularities means that we have to decide how to treat them to get a well-defined result. There are several ways to do this, and each has a physical interpretation. If we integrate over the “time" component k_0 first, we get

$$D(z) = \frac{-1}{(2\pi)^4}\int d^3k e^{i\vec{k}.\vec{z}}\int dk_0 \frac{e^{-ik_0z_0}}{k_0{}^2 - \kappa^2} \tag{14}$$

The integrand of the k_0 integral has two simple poles at the points $k_0 = \pm\kappa$. The integral can be easily evaluated by the method of contour integration wherein we consider k_0 to be a complex variable. First we select a suitable contour for the purpose. Note that the poles of this integral are both real. This means that the integral is ambiguous - it can be assigned any of several possible values depending on how we choose to evaluate it. Green’s functions with different behaviour are obtained by choosing different contours of integration relative to poles. [Or equivalently we can shift the poles away from the real axis by assigning a small imaginary part to k_0. In the end we take the limit such that the poles return to the real axis.] **[See Figure 12.8 Jackson Edition 2]**



Let us choose the contour to be shifted slightly above the real axis. The open contours from $(-\infty,\infty)$ may be closed at infinity with a semicircle in the lower or upper half-plane, depending on the sign of z_0 in the exponential. For $z_0 > 0$, the exponential, $e^{-ik_0 z_0}$, tends to infinity in the upper half plane and exponentially to zero in the lower half plane. Therefore in order to use the residue theorem to evaluate the integral the contour must be closed in the lower half-plane. Similarly, for $z_0 < 0$, the contour has to be closed in the upper half plane. Thus for $z_0 < 0$ the two singularities then lie outside the contour, since they have been shifted slightly above the real axis, and the integral is zero. Regarding k_0 as a complex variable the integral over the closed contour can be written as

$$\oint_\Gamma dk_0 \frac{e^{-iz_0 k_0}}{z_0^{\,2} - \kappa^2} = \int_{-\infty}^{\infty} dk_0 \frac{e^{-iz_0 k_0}}{z_0^{\,2} - \kappa^2} = \int_C dk_0 \frac{e^{-iz_0 k_0}}{z_0^{\,2} - \kappa^2}$$

As noted the integral over C clearly vanishes for $z_0>0$. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} dk_0 \frac{e^{-iz_0 k_0}}{z_0^{\,2} - \kappa^2} &= \oint_\Gamma dk_0 \frac{e^{-iz_0 k_0}}{z_0^{\,2} - \kappa^2} \\ &= (\lim \varepsilon \to 0)[(-2\pi i)\,\mathrm{Res}\,\frac{e^{-iz_0 k_0}}{\{k_0 - (\kappa - i\varepsilon)\}\{k_0 + (\kappa + i\varepsilon)\}}] \qquad (15) \\ &= -2\pi i\Big(\frac{e^{-iz_0\kappa}}{2\kappa} + \frac{e^{iz_0\kappa}}{-2\kappa}\Big) = -2\pi \frac{\sin(\kappa z_0)}{\kappa} \end{aligned}$$

We can then write the Green's function (14) as

$$D(z) = \frac{\theta(z_0)}{(2\pi)^3} \int d^3k e^{i\vec{k}.\vec{z}} \frac{\sin(\kappa z_0)}{\kappa}$$

$\theta(x)$ is the usual theta function which has value unity for its argument $x>0$ and zero for $x<0$. We now do the integral over d^3k by changing over to spherical polar coordinates so that

$$\begin{aligned} D(z) &= \frac{\theta(z_0)}{(2\pi)^3} \int_0^\infty \kappa^2 d\kappa \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi e^{i\kappa R\cos(\theta)} \frac{\sin(z_0\kappa)}{\kappa} \\ &= \frac{\theta(z_0)}{(2\pi)^2} \int_0^\infty d\kappa\kappa \sin(z_0\kappa) \int_0^\pi \sin(\theta) d\theta e^{i\kappa R\cos(\theta)} \end{aligned}$$

Here $R = |\vec{z}| = |\vec{x} - \vec{x}'|$. The integration over $d\theta$ gives

$$\int_0^{\pi} \sin(\theta) d\theta e^{i\kappa R\cos(\theta)} = \frac{2}{\kappa R}\sin(\kappa R)$$

so that

$$D(z) = \frac{\theta(z_0)}{2\pi^2 R}\int_0^{\infty} d\kappa \sin(z_0\kappa)\sin(R\kappa) = \frac{\theta(z_0)}{4\pi^2 R}\int_{-\infty}^{\infty} d\kappa \sin(z_0\kappa)\sin(R\kappa)$$

Using the representation of sine function in terms of exponential, the last integral can be rewritten as

$$D(z) = \frac{\theta(z_0)}{8\pi^2 R}\int_{-\infty}^{\infty} d\kappa [e^{i(z_0 - R)\kappa} - e^{i(z_0 + R)\kappa}]$$

On using the integral representation of the delta function:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikr}$$

we have

$$D_r(z) = \frac{\theta(x_0 - x'_0)}{4\pi R}[\delta(x_0 - x'_0 - R) - \delta(x_0 - x'_0 + R)]$$

Here we have reverted to the original variables:

$$z = x - x' \Rightarrow z_0 = x_0 - x'_0, |\vec{z}| = |\vec{x} - \vec{x}'| = R$$

Finally we notice that because of the theta function, $x_0 > x'_0 \Rightarrow x_0 + R > x'_0$, and therefore the second delta function does not contribute, so that

$$D_r(x - x') = \frac{\theta(x_0 - x'_0)}{4\pi R}\delta(x_0 - x'_0 - R) \qquad (16)$$

The suffix "*r*" on the Green's function refers to "retarded". This Green's function is called the *retarded or causal* Green's function because the source point time x'_0 is always *earlier* than the observation point time, x_0. This is in accordance with *causality*, one of the fundamental principles of physics and states that the cause always precedes the effect.

Had we chosen the contour to be slightly below the real axis, the result would have been the *advanced Green's function*

$$D_a(x - x') = \frac{\theta[-(x_0 - x'_0)]}{4\pi R}\delta(x_0 - x'_0 + R) \qquad (17)$$

For whatever it is worth, the Green's function can be put in an explicitly covariant form by making use of the delta function identity

$$\delta(x^2 - a^2) = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)]$$

so that

$$\delta[(x-x')^2] = \delta[(x_0 - x'_0)^2 - |\vec{x} - \vec{x}'|^2] = \frac{1}{2R}[\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)]$$

Multiplying by the theta function, we have

$$\theta(x_0 - x'_0)\delta[(x-x')^2] = \frac{1}{2R}\theta(x_0 - x'_0)[\delta(x_0 - x'_0 - R) + \delta(x_0 - x'_0 + R)]$$
$$= \frac{1}{2R}\theta(x_0 - x'_0)[\delta(x_0 - x'_0 - R)]$$

The second delta function does not contribute as we have already seen. Hence

$$D_r(x-x') = \frac{1}{2\pi}\theta(x_0 - x'_0)\delta[(x-x')^2] \tag{18}$$

Similarly

$$D_a(x-x') = \frac{1}{2\pi}\theta(x'_0 - x_0)\delta[(x-x')^2] \tag{19}$$

The theta functions though seem to be noninvariant are actually invariant. Because of the delta function the Greens' functions contribute only on the light cone, D_r on the forward light cone and D_a only on the backward light cone of the source point.

The solution of the wave equation (3) can now be written in the form (5)

$$A^\alpha(x) = A^\alpha_{in}(x) + \mu_0 \int d^4x' D_r(x-x') J^\alpha(x') \tag{20}$$

$$A^\alpha(x) = A^\alpha_{out}(x) + \mu_0 \int d^4x' D_a(x-x') J^\alpha(x') \tag{21}$$

where $A^\alpha_{in}(x)$ and $A^\alpha_{out}(x)$ are solutions of the homogeneous wave equation $\Box A^\alpha = 0$. In equation (16) retarded Green's function is used. In the limit $x_0 \to -\infty$, the integral over sources vanishes, assuming the sources are localized in space and time, because of the retarded nature of Green's function. We see that the free-field potential $A^\alpha_{in}(x)$ has the interpretation of "incident" or "incoming" potential specified at $x_0 \to -\infty$. Similarly, in equation (17), with the advanced Green's function, the homogeneous solution $A^\alpha_{out}(x)$ is the asymptotic "outgoing" potential, specified for $x_0 \to +\infty$. The *radiation fields* are defined as the difference between the outgoing and the incoming fields. Their four-vector potential is

$$A^\alpha_{rad} = A^\alpha_{out} - A^\alpha_{in}(x) = \mu_0 \int d^4x' D(x-x') J^\alpha(x')$$

where

$$D(z) = D_r(z) - D_a(z)$$

26.3 Liénard-Wiechert Potentials

We will now stick to the retarded Green's function, the one we need to calculate the fields for a source consisting of a charge moving along a given trajectory. We begin with the expression for the four-potential in terms of the Green's function [see equation (7)]:

$$A^\alpha(x) = \mu_0 \int d^4x' D_r(x-x') J^\alpha(x') \qquad (22)$$

with the Green's function $D_r(x-x')$ given by equation (16) or its covariant form equation (18). For an arbitrarily moving point charge *e*, let its position at any time *t* be given by $\vec{r}(t)$ and velocity by $\vec{v}(t)$. The charge and current densities at any time *t* can be represented by

$$\begin{aligned} \rho(\vec{x}', t) &= e\delta[\vec{x}' - \vec{r}(t)] \\ \vec{J}(\vec{x}', t) &= e\vec{v}\,\delta[\vec{x}' - \vec{r}(t)] \end{aligned} \qquad (23)$$

If $r^\alpha(\tau)$ is the position four vector, $V^\alpha(\tau)$ the velocity four-vector of the particle as a function of its proper time τ, $V^\alpha = (V^0, \vec{V}) = (\gamma c, \gamma\vec{v})$ [γ is the Lorentz boost factor $1/\sqrt{1-v^2/c^2}$], then the four-current density $J^\alpha = (c\rho, \vec{J})$ can be written as

$$J^\alpha(x') = ec\int d\tau V^\alpha(\tau)\delta^{(4)}([x' - r(\tau)]) \qquad (24)$$

[$J^\alpha(x') = ec\int d\tau V^\alpha(\tau)\delta^{(4)}([x' - r(\tau)]) = ec\int d\tau V^\alpha(\tau)\delta([\vec{x}' - \vec{r}(\tau)])\delta(ct' - ct)$

$= ec\int \frac{dt}{\gamma} V^\alpha(\tau)\delta([\vec{x}' - \vec{r}(\tau)])\frac{1}{c}\delta(t'-t) = e\{c, \vec{v}\}\delta\{(\vec{x}' - \vec{r}(t)\}$]

To do the integral involved in the expression (22) for A^α, we need the explicitly covariant form of the Green's function, equation (18). Using equation (24) for J^α and equation (18) for $D_r(x-x')$ into equation (22), we have

$$\begin{aligned} A^\alpha(x) &= \mu_0 \int d^4x' \frac{1}{2\pi}\theta(x_0 - x'_0)\delta[(x-x')^2] ec\int d\tau V^\alpha(\tau)\delta^{(4)}([x' - r(\tau)]) \\ &= \frac{ec\mu_0}{2\pi}\int d\tau V^\alpha(\tau)\theta(x_0 - r_0(\tau))\delta\{[(x - r(\tau)]^2\} \end{aligned} \qquad (25)$$

Here we have made use of the 4-dimensional delta function $\delta^{(4)}([x'-r(\tau)])$ to do the d^4x' integration and replace the four vector x' by the four vector $r(\tau)$. The remaining integral over the charged particle's proper time gives a contribution at $\tau=\tau_0$, where τ_0 is defined by the light cone condition

$$[(x-r(\tau)]^2 = 0 \tag{26}$$

and the retardation requirement $x_0 > r_0(\tau_0)$. The significance of these conditions can be seen from the diagram **[See Figure]**. The figure shows the light cone of the observation point. The delta function $\delta[(x-r(\tau)]^2$ ensures that the trajectory of the charge must lie on the light cone and the theta function $\theta(x_0 - r_0(\tau))$ ensures that it lies on the backward light cone. Since the velocity of the charge has to be less than the velocity of light, the tangent to the trajectory must make an angle of more than 45^0 at every point along it. Hence the world line of the particle intersects the light cone only at two points, one on the forward light cone and the other on the backward line cone. The backward light cone point is the only one that contributes because of the theta function.

To evaluate the integral in (25) we use the well-known relation for the delta-function

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{\left|(\frac{df}{dx})_{x=x_i}\right|} \tag{27}$$

where the points $x = x_i$ are the zeros of $f(x)$, assumed to be simple. The function $f(x)$ is assumed to be smooth so that the derivative exists.

[Consider $\int_{-\infty}^{\infty} \delta[f(x)]g(x)dx$. The integral will get contribution from all points where $f(x) = 0$. Let $x = x_i$ be one such point. Expand $f(x)$ around x_i :

$$f(x) = f'(x_i)(x-x_i) + .. \Rightarrow \delta[f(x)] = \delta[f'(x_i)(x-x_i)] = \frac{1}{|f'(x_i)|}\delta(x-x_i)$$

Since similar contribution comes from each zero, $\delta[f(x)] \equiv \sum_i \frac{\delta(x-x_i)}{\left|(\frac{df}{dx})_{x=x_i}\right|}$].

To evaluate equation (25) we make use of equation (27) and obtain

$$A^\alpha(x) = \frac{\mu_0 ec}{2\pi}\int d\tau V^\alpha(\tau)\theta(x_0 - r_0(\tau))\delta\{[(x-r(\tau)]^2\}$$

$$= \frac{\mu_0 ec}{2\pi}\int d\tau V^\alpha(\tau)\theta(x_0 - r_0(\tau))\frac{\delta(\tau-\tau_0)}{\left|\frac{d}{d\tau}[x-r(\tau)]^2\right|_{\tau=\tau_0}}$$

Now

$$\frac{d}{d\tau}[(x-r(\tau)]^2 = 2[(x-r(\tau)]_\beta \frac{d}{d\tau}[(x-r(\tau)]^\beta = -2[(x-r(\tau)]_\beta V^\beta(\tau)$$

so that

$$A^\alpha(x) = \frac{\mu_0 ec}{2\pi}\int d\tau V^\alpha(\tau)\theta(x_0 - r_0(\tau))\frac{\delta(\tau-\tau_0)}{\left|-2[(x-r(\tau)]_\beta V^\beta(\tau)\right|_{\tau=\tau_0}}$$

or

$$A^\alpha(x) = \frac{\mu_0 c}{4\pi}\frac{eV^\alpha(\tau)}{V.(x-r(\tau))}\bigg|_{\tau=\tau_0} \tag{28}$$

These potentials in their various forms are called *Liénard-Wiechert potentials*. We can also put them into more familiar non-covariant form. Now τ_0 is the solution of light-cone condition (7) and implies $x_0 - r_0(\tau_0) = |\vec{x} - \vec{r}(\tau_0)| = R$. Thus

$$V.(x-r) = V_0[x_0 - r_0(\tau_0)] - \vec{V}.[\vec{x} - \vec{r}(\tau_0)] = \gamma cR - \gamma\vec{v}.\vec{R} = \gamma cR(1-\vec{\beta}.\hat{n}) \tag{29}$$

where , as usual, $\vec{\beta} = \vec{v}/c$ and $\hat{n} = \frac{\vec{R}}{R}$ is a unit vector in the direction of $\vec{R}$. Using this relation in equation (28) above, the usual scalar and vector potentials are

$$\Phi(\vec{x},t) = \frac{\mu_0 c^2}{4\pi}\left[\frac{e}{R(1-\vec{\beta}.\hat{n})}\right]_{ret,}, \qquad \vec{A}(\vec{x},t) = \frac{\mu_0 c}{4\pi}\left[\frac{e\vec{\beta}}{R(1-\vec{\beta}.\hat{n})}\right]_{ret} \tag{30}$$

The square bracket with the subscript “ret” means the entire expression must be evaluated at the retarded time $r_0(\tau_0) = x_0 - R$. For nonrelativistic motion ($|\vec{\beta}| << 1$) these expressions evidently reduce to the well-known forms for the scalar and vector potentials.

26.4 The Electromagnetic Field Tensor

The electromagnetic fields, $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, can be obtained from any of the forms for A^α that we have derived above, but it is more convenient to begin from the form (25) that we started with. The derivative ∂^α acts on the space-time coordinate *x*, which appears inside the theta and delta functions. Let us look at one of the terms $\partial^\alpha A^\beta$ first:

$$\partial^\alpha A^\beta(x) = \frac{ec\mu_0}{2\pi}\int d\tau U^\beta(\tau)\partial^\alpha\{\theta(x_0 - r_0(\tau))\delta\{[(x-r(\tau)]^2\}\}$$

The derivative of the theta function is the delta function. So $\partial^\alpha\{\theta(x_0 - r_0(\tau))\}$=0 for α=1,2 and 3, and $\partial^0\{\theta(x_0 - r_0(\tau))\} = \delta(x_0 - r_0(\tau))$. As a result the delta function is constrained to be $\delta\{[(x-r(\tau)]^2\} = \delta\{[x_0 - r_0(\tau)]^2 - R^2\} = \delta(-R^2)$ and so does not contribute except at R=0. The value of the fields at the position of the point is a problematic thing anyway and therefore we exclude the point R=0 from consideration. Thus except at R=0 the term becomes

$$\partial^\alpha A^\beta(x) = \frac{ec\mu_0}{4\pi}\int d\tau V^\beta(\tau)\theta(x_0 - r_0(\tau))\partial^\alpha\delta\{[(x-r(\tau)]^2\} \qquad (31)$$

Doing the indicated differentiation, we have { $f = [x-r(\tau)]^2 = [x-r(\tau)]^\beta[x-r(\tau)]_\beta$ }

$$\partial^\alpha\delta[f] = \partial^\alpha f\frac{d}{df}\delta[f] = \partial^\alpha f\frac{d}{d\tau}\delta[f]\frac{d\tau}{df} = \frac{\partial^\alpha f}{(\frac{df}{d\tau})}\frac{d}{d\tau}\delta[f]$$

Further

$$\partial^\alpha f = 2[x-r(\tau)]^\alpha, \qquad \frac{df}{d\tau} = -2[x-r(\tau)]^\beta\frac{dr_\beta}{d\tau} = -2[V.(x-r)]$$

so that

$$\partial^\alpha\delta[f] = -\frac{(x-r)^\alpha}{V.(x-r)}\frac{d}{d\tau}\delta[f]$$

Inserting back into equation (31) we have

$$\partial^\alpha A^\beta(x) = -\frac{ec\mu_0}{4\pi}\int d\tau V^\beta(\tau)\theta(x_0 - r_0(\tau))\frac{(x-r)^\alpha}{V.(x-r)}\frac{d}{d\tau}\delta\{[(x-r(\tau)]^2\}$$

Now we do the $d\tau$ integration by parts.

$$\partial^\alpha A^\beta(x) = \frac{ec\mu_0}{4\pi}\int d\tau\frac{d}{d\tau}\{V^\beta(\tau)\theta(x_0 - r_0(\tau))\frac{(x-r)^\alpha}{V.(x-r)}\}\delta\{[(x-r(\tau)]^2\}$$

As before, the derivative of the theta function yield delta function which constrains the other delta function to $\delta(-R^2)$. Since the origin is already excluded, this term does not contribute. So we have

$$\partial^\alpha A^\beta(x) = \frac{ec\mu_0}{4\pi}\int d\tau\frac{d}{d\tau}\{V^\beta(\tau)\frac{(x-r)^\alpha}{V.(x-r)}\}\theta(x_0 - r_0(\tau))\delta\{[(x-r(\tau)]^2\}$$

This equation has exactly the same form as equation (25) with the replacement $V^{\alpha} \rightarrow \frac{d}{d\tau}\{\frac{(x-r)^{\alpha}V^{\beta}(\tau)}{V.(x-r)}\}$. On following exactly the same footsteps as we followed after equation (25) to arrive at equation (28), we now arrive at the expression

$$\partial^{\alpha}A^{\beta}(x) = \frac{e}{V.(x-r)}\frac{d}{d\tau}\{\frac{(x-r)^{\alpha}V^{\beta}(\tau)}{V.(x-r)}\} \tag{32}$$

Treating the other term $\partial^{\beta}A^{\alpha}$ identically, we finally get

$$F^{\alpha\beta}(x) = \frac{c\mu_0}{4\pi}\frac{e}{V.(x-r)}\frac{d}{d\tau}\{\frac{(x-r)^{\alpha}V^{\beta}(\tau)-(x-r)^{\beta}V^{\alpha}(\tau)}{V.(x-r)}\} \tag{33}$$

Here r^{α} and v^{α} are functions of τ. After differentiation, the whole expression is to be evaluated at the retarded time τ_0 .

26.5 The Electric and Magnetic Fields

This result is beautifully covariant but not particularly transparent. We are more used to the explicit $\vec{E}$ and $\vec{B}$ fields than the covariant form. In any case the power radiated is given in terms of the Poynting vector which is simply expressed in terms of the $\vec{E}$ and $\vec{B}$ fields. Thus from a practical point of view we would like to get explicit expressions for the $\vec{E}$ and $\vec{B}$ fields in terms of the velocity and acceleration of the particle. To obtain the results in non-covariant, but more familiar form, we do the differentiation in (33):

$$\begin{aligned} F^{\alpha\beta}(x) = \frac{c\mu_0}{4\pi}[&\frac{e}{\{V.(x-r)\}^2}\frac{d}{d\tau}\{(x-r)^{\alpha}V^{\beta}(\tau)-(x-r)^{\beta}V^{\alpha}(\tau)\} \\ &-\frac{e}{\{V.(x-r)\}^2}\{(x-r)^{\alpha}V^{\beta}(\tau)-(x-r)^{\beta}V^{\alpha}(\tau)\}\frac{d}{d\tau}V.(x-r)] \end{aligned} \tag{34}$$

Now

$$(x-r)^{\alpha} = (R, R\hat{n}), \qquad V^{\alpha} = (\gamma c, \gamma c\vec{\beta})$$

$$\frac{dV^0}{d\tau} = \gamma\frac{d}{dt}(\gamma c) = c\gamma\frac{d}{dt}(1-\frac{v^2}{c^2})^{-1/2} = c\gamma^4(\vec{\beta}.\dot{\vec{\beta}})$$

$$\frac{d\vec{V}}{d\tau} = \gamma\frac{d}{dt}(\gamma\vec{v}) = \vec{v}\gamma\frac{d\gamma}{dt}+\gamma^2\frac{d\vec{v}}{dt} = c\gamma^4(\vec{\beta}.\dot{\vec{\beta}})\beta + c\gamma^2\dot{\vec{\beta}}$$

$$\begin{aligned}\frac{d}{d\tau}[V.(x-r(\tau))] = \frac{d}{d\tau}[V^{\alpha}(x-r(\tau))_{\alpha}] &= \frac{dV^{\alpha}}{d\tau}(x-r)_{\alpha} - V^{\alpha}V_{\alpha} \\ &= \frac{dV^{\alpha}}{d\tau}(x-r)_{\alpha} - c^2\end{aligned}$$

$$(x-r)_\alpha \frac{dV^\alpha}{d\tau} = R\frac{dV^0}{d\tau} - R\hat{n}.\frac{d\vec{V}}{d\tau} = Rc\gamma^4(\vec{\beta}.\dot{\vec{\beta}})(1-\hat{n}.\vec{\beta}) - Rc\gamma^2(\hat{n}.\dot{\vec{\beta}})$$

Now substitute all these expressions into equation (34) and find its various components ($\vec{E}_i / c = F^{i0}, B_1 = F^{32}$ etc.). After a very tedious but otherwise straightforward calculation, we obtain

$$\vec{E}(\vec{x},t) = \frac{e}{4\pi\varepsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}.\hat{n})^3 R^2}\right]_{ret} + \frac{e}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times\{(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}\}}{(1-\vec{\beta}.\hat{n})^3 R}\right]_{ret} \tag{35}$$

$$\vec{B} = \frac{1}{c}[\hat{n}\times\vec{E}]_{ret} \tag{36}$$

Expression (35) is written in such a way that the first term is independent of the acceleration and depends only on its velocity. This is called *velocity field*, while the second term is called the *acceleration field*. Also the first term falls off as ~$1/R^2$, while the second term falls off as ~$1/R$. The first term is clearly the *usual static field*:

$$\vec{E} \approx e\frac{\hat{n}}{R^2}\,.$$

The accelerations fields are transverse and fall off as ~$1/R$ – they are the typical *radiation fields*. If you like, the first terms are the ``near" and ``intermediate" fields and the second is the complete ``far" field; only the far field is produced by the acceleration of a charge. Only this field contributes to a net radiation of energy and momentum away from the charge.

Summary

1. *The general nature of the wave equation is described.*
2. *The green's method of solving inhomogeneous differential equations is discussed.*
3. *Green's function for the wave equation in a covariant form is obtained and then solved by using the method of Fourier transforms.*
4. *The four-potential produced by the motion of a charged particle, called Liénard-Wiechert potentials, is obtained from the solution of the wave equation.*
6. *The electromagnetic field tensor is obtained from these potentials.*
7. *The explicit expressions for the electric and the magnetic field are obtained from the field tensor.*